Higher-order Logic Programming with $\lambda$Kanren

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We present $\lambda$Kanren, a new member of the Kanren family [2] that is inspired by $\lambda$Prolog [5]. With a shallow embedding implementation, the term language of $\lambda$Kanren is represented by the functions and macros of its host language. As a higher-order logic programming language, $\lambda$Kanren is extended with a subset of higher-order hereditary Harrop formulas [7].

1 INTRODUCTION

$\lambda$Kanren introduces four new operators to $\mu$Kanren [3]: tie, app, assume-rel, and all. The operators tie and app create binding structures. In addition, the $\equiv$ operator recognizes $\alpha\beta$-conversions between binding structures. The assume-rel and all operators enable more expressive reasoning with Hereditary Harrop formulas [6]. To demonstrate $\lambda$Kanren’s increment to $\mu$Kanren, we first review the two forms of logic, fohc and hohh, behind these two languages.

$\mu$Kanren implements First-order Horn clause (fohc) [1]. The grammar of Horn clause is shown in Fig 1. We say $\mu$Kanren is first-order, as its unification algorithm identifies only structural equivalence. As an example that illustrates the correspondence between $\mu$Kanren definitions and fohc formulas, consider the relation append$^o$.

\[
\text{(defrel (append$^o$ xs ys zs))}
\]

\[
\text{(cond$^c$)}
\]

\[
[(\equiv \text{nil} \ \text{xs}) \ (\equiv \text{ys} \ \text{zs})]
\]

\[
[(\text{fresh} \ (a \ d \ r))
\]

\[
(\equiv \ '(,a \ ,d) \ \text{xs})
\]

\[
(\text{append$^o$} \ d \ \text{ys} \ r)
\]

\[
(\equiv \ '('(,a \ ,r) \ \text{zs}))]]
\]

$D$ formulas of fohc. In $\mu$Kanren, a defrel introduces a $D$ formula. For example, the append$^o$ definition corresponds to this $D$ formula,

\[
\forall \text{xs} \ \forall \text{ys} \ \forall \text{zs} \ (\equiv \text{xs} \ \text{nil}) \land (\equiv \text{ys} \ \text{zs})
\]

\[
\lor \exists a \ \exists d \ \exists r (\equiv \text{xs}'(,a,d)) \land (\text{append$^o$} \ d \ \text{ys} \ r) \land (\equiv \ '(,a,r) \ \text{zs})
\]

\[
\triangleright (\text{append$^o$} \ \text{xs} \ \text{ys} \ \text{zs}).
\]

Here append$^o$ and $\equiv$ both build atomic formulas. For example, (append$^o$ xs ys zs) and (\equiv \xs \nil) are atomic formulas.

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Fig. 1. Horn Clause Formulas

\[
\begin{align*}
\text{Goals} & \quad G ::= A | G \land G | G \lor G | \exists x \ G \\
\text{Definitions} & \quad D ::= A | G \supset D | D \land D | \forall x \ D \\
\text{Atomic Formulas} & \quad A
\end{align*}
\]

Fig. 2. Hereditary Harrop Formulas

\[
\begin{align*}
\text{Goals} & \quad G ::= A | G \land G | G \lor G | \exists x \ G | D \supset G | \forall x \ G \\
\text{Definitions} & \quad D ::= A | G \supset D | D \land D | \forall x \ D \\
\text{Atomic Formulas} & \quad A
\end{align*}
\]

\textit{G formulas of fohc.} In \(\mu\text{Kanren},\) a \texttt{run} query contains a \textit{G formula}, e.g.,

\begin{verbatim}
(run 1
  (fresh (xs)
   (append^\circ xs '~(1 2) ~(1 2))))
\end{verbatim}

is formulated as

\[
\exists xs (append^\circ xs ~(1 2) ~(1 2)).
\]

\textit{Formulas of hohh.} \(\lambda\text{Prolog}\) implements a more expressive logic, \textit{higher-order hereditary Harrop formulas (hohh)} [6]. Shown in Figure 2, Hereditary Harrop formulas extend \(G\) formulas with implicational goals and forall-quantification. Also, with higher-order unification, the unification algorithm of \(\lambda\text{Prolog}\) identifies \(\alpha\beta\)-equivalence between binding structures (that are absent in \(\mu\text{Kanren}\)).

This paper presents \(\lambda\text{Kanren}.\) \(\lambda\text{Kanren}\) implements implicational goals and forall-quantification with two new operators, \texttt{assume-rel} and \texttt{all}, respectively. Also, \(\lambda\text{Kanren}\) incorporates higher-order pattern unification [4] for the binding structures (that are created by another two new operators, \texttt{tie} and \texttt{app}).

The rest of this paper demonstrates the uses of these four operators and their implementation details when appropriate. Our implementation of \(\lambda\text{Kanren}\) is available at https://github.com/mvcccccc/MK2020.

\section{Higher-Order Unification}

This section shows the power of higher-order pattern unification. By adapting Miller [4]'s unification algorithm, \(\lambda\text{Kanren}\) is equipped with two new operators: \texttt{tie} and \texttt{app}. \texttt{tie} expressions are abstractions and \texttt{app} is the shorthand for application. The \texttt{≡} operator in \(\lambda\text{Kanren}\) identifies \(\alpha\beta\)-equivalence between terms that involve \texttt{tie} and \texttt{app}.

Consider the following example that demonstrates \(\alpha\)-equivalence. This example, metaphorically, tests the equivalence between \(\lambda (a \ b) (a \ b)\) and \(\lambda (x \ y) (x \ y)\).

\begin{verbatim}
> (run 1 q
  (≡ (tie (a b) (app a b))
    (tie (x y) (app x y)))))
'(≡)
\end{verbatim}

tie is implemented as the following macro. It takes a list of variable names and a term. It then creates a \texttt{Tie} structure that is internally used for curried binders. Hereafter, we call a variable that is introduced by \texttt{fresh} a \textit{unification variable} and a variable that is introduced by \texttt{tie} a \textit{binding variable}. 

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(\text{\texttt{define-syntactix tie}}
(\text{\texttt{syntax-rules (}}
  \[(\_ \_ \text{t} \_ \text{body})\]
  \[(\_ \_ \text{x}_0 \_ \_ \ldots \_ \text{body})\]
  \[\_ \_ \text{x}_0 \_\text{Var} \_ \text{x}_0\_\]
  \[\_ \_ \text{x}_0 \_\text{tie} \_ \text{x}_0 \_\ldots \_ \text{body} \_\]]\)
)

\text{\texttt{app}} is implemented as the following macro that elaborates a list of terms to an App structure that is internally used for curried applications.

(\text{\texttt{define-syntactix app}}
(\text{\texttt{syntax-rules (}}
  \[(\_ \_ \text{rator} \_ \_ \text{rand} \_ \_ \_ \ldots \_ \text{rand})\]
  \[(\_ \_ \text{rator} \_ \_ \text{rand}_0 \_ \_ \_ \ldots \_ \text{rand} \_ \_ \_ \ldots \_ \text{rand})\]
  \[\_ \_ \text{app} \_\_ \text{rator} \_\_ \text{rand}_0 \_\_ \_ \ldots \_ \text{rand} \_ \_ \_ \ldots \_ \text{rand} \_\_\]]\)
)

Next, consider the following example that queries for two instantiations of \(f\). This example demonstrates (1) \(\beta\)-conversions during unification and (2) how binding structures are reified.

\begin{verbatim}
> (run 2 f
  (== (tie (a b) (app a b))
       (tie (x y) (app f x y))))
'((tie (_0) (tie (_1) (app _0 _1))))
\end{verbatim}

There is only one instantiation: \(f\) is a function (a Tie structure) of two inputs and \(f\) outputs an application form (a App structure) that applies its first input on the second one.

The internal structures, Tie and App, are reified as tagged lists. These tagged lists reflect their corresponded user interfaces, tie and app. During reification, binding variables and unification variables are both converted to underscore-digit symbols.

The power of higher-order unification, however, comes in with limits. To ensure decidability, \(\beta\)-conversion in Miller [4]'s algorithm restricts application forms: when the operator of an app is a unification variable, its operands must be distinct binding variables, otherwise unification fails. For example, the following query has no solution because the operands, the two bs, of the unification variable \(f\) are not distinct. In this case, with \(f\) being a function of two input bs, we cannot decide which b takes control.

\begin{verbatim}
> (run 1 f
  (== (tie (a) a)
       (tie (b) (app f b b))))
'()
\end{verbatim}

To enforce this restriction, Miller [4]'s algorithm imposes another restriction on variable scopes: the instantiation of a unification variable may only contain its \textit{visible} binding variables. A binding variable \(x\) is visible to a unification variable \(q\) if the introduction of \(x\) lexically precedes that of \(q\). Given the following example, it seems that \(q\) can be instantiated by \(y\). Unfortunately, \(y\) is not visible to \(q\) and the query has no solution.

\begin{verbatim}
> (run 1 q
  (== (tie (a b) (app a b))
       (tie (x y) (app x q))))
'()
\end{verbatim}
In our implementation, the unifier extends higher-order pattern unification and adapts it for conventional miniKanren programming style. In Miller [4]'s algorithm, a unification variable must be an operator of an application form. The operands of the application must be distinct binding variables. That means, Miller [4]'s algorithm reports unsolvability when a unification variable is simply unified against a constant. (In fact, constants are not in the term language of Miller [4]'s algorithm).

In miniKanren, we often unify a single unification variable and a constant. So, it is compelling that we extend the term language and the unification variable, as we have done in our implementation.

3 IMPLICATIONAL GOALS

This section introduces the assume-rel operator that implements implicational goals ($D \supset G$). An assume-rel operator takes two inputs: (1) the hypothesis in the form of a $D$ formula and (2) the goal in the form of a $G$ formula. The assume-rel operator then uses the hypothesis as a fact and moves on to the goal.

Implementing assume-rel is subtle with shallow embedding. Because the definitions of $\lambda$Kanren are kept in the run-time environment of its host language, extending these definitions requires updating the run-time environment. This problem is illustrated in the following example, liberally adapted from Miller and Nadathur [5, p. 80].

```scheme
(defrel (taken name class)
    (conde
        [($(= 'Josh name) $(= 'B521 class))]
        [($(= 'Josh name) $(= 'B522 class))])
(defrel (pl-major name)
    (taken name 'B521)
    (taken name 'B523)
    (taken name 'B522))
```

One may complete pl-major after taking three classes: B521, B522, and B523. And Josh currently has taken B521 and B522. In the following query, the assume-rel operator extends the definition of taken with (taken 'Josh q) and then moves on to the goal (pl-major 'Josh).

```scheme
> (run 1 q
  (assume-rel [(taken name class)
               (=(= 'Josh name) (=(= 'q class))]
               (pl-major 'Josh)))
'(B523)
```

From the implementation aspect, because the host language is lexically scoped, the definition of pl-major is fixed. This means that, the free variable in the definition of pl-major, namely taken, always uses the original definition of Josh taking B521 and B522. To extend definitions on the fly, we need to create dynamic scope so that the free variables may use the latest, updated definitions.

Our approach is to add an extra layer between $\lambda$Kanren and the host language (Racket). This extra layer redirects function definitions.

We introduce two global maps, name->idx and idx->def. Each defrel extends these two maps by creating a new index, putting the name-idx pair and the idx-def pair in the two maps respectively. The idx->def map is global. And the name->idx map is threaded through during the execution of a query (an invocation of a run).
To invoke a definition, one follows name->idx and idx->def, i.e., first retrieving the index using name->idx and then getting the definition using idx->def. For example, the user interface

\[(\text{pl-major}'\text{Josh})\]

is macro-expanded to

\[(((\text{cdr} (\text{assv}) \text{idx->def}) (\text{cdr} (\text{assv} \text{name->idx} '\text{pl-major})))) '\text{Josh}).\]

When an assume-rel operator is invoked, the two maps are extended again: (1) a new index is created; (2) idx->def contains the pair of the new index and the extended function; and (3) name->idx now has a new pair of the definition name and the new index, this new pair shadows the previous one.

In the previous example, let’s use \(t_1\) for the taken definition that knows Josh has taken B521 and B522, use \(t_2\) for the extended taken (where we assume-rel Josh has taken B523), and use \(p\) for the definition of pl-major. With taken and pl-major first defined, name->idx is \((\text{pl-major} . 2) (\text{taken} . 1))\) and idx->body is \(((2 . p) (1 . t_1))\). Then, after assuming \((\text{taken} '\text{Josh} q)\), the query \((\text{pl-major} '\text{Josh})\) runs in an updated environment where name->idx is \((\text{taken} . 3) (\text{pl-major} . 2) (\text{taken} . 1))\) and idx->body is \(((3 . t_2) (2 . p) (1 . t_1))\). The more recent pair in name->idx shadows the previous one. And therefore, when taken is invoked, we use \(t_2\).

Many interesting examples only make hypothesis on atomic formulas. And thus we provide the assume operator that is a shorter version of the assume-rel operator. Instead of any D formula, the assume operator only takes an atomic hypothesis.

As an example, we define the eq relation to be reflexive, transitive, and symmetric as follows.

\[
\begin{align*}
\text{(defrel } \text{eq } x y) \\
\text{(cond' } \\
\quad [\text{=} x y)] \\
\quad [(\text{fresh } z) \\
\quad \quad (\text{eq } x z) \\
\quad \quad (\text{eq } z y))] \\
\quad [(\text{eq } y x))]
\end{align*}
\]

Obviously apple and orange are by no means eq. In fact, the following query does not terminate in a naive \(\mu\)Kanren implementation because the third cond' line is very recursive.

\[
> \text{(run 1 q} \\
\text{ (eq 'apple 'orange))}
\]
Using `assume`, we may temporarily extend the definition of `eq` as follows.

```prolog
> (run 5 q
   (assume (eq 'orange 'apple)
     (assume (eq 'orange 'dog)
       (eq 'orange q))))
'(dog apple orange orange dog)
```

Because λKanren runs backward, as in the following, the hypothesis can be inferred as well.

```prolog
> (run 1 q
   (assume (eq 'orange q)
     (eq 'apple 'orange))))
'(apple)
```

4 FORALL-QUANTIFICATION

This section introduces the `all` operator ($\forall x G$) that takes a list of symbols and a goal. These symbols are used to create special variables that are virtually constants (eigenvariables).

Continuing with `taken` and `pl-major`, we create a random person $x$ using the `all` operator.

```prolog
> (run 1 q
   (all (x)
     (assume - rel (taken x 'B521)
      (assume - rel (taken x 'B522)
       (assume - rel (taken x 'B523)
        (pl - major x))))))
'(_0)
```

Like the `fresh` operator, the `all` operator creates a new variable in the scope. Unlike the `fresh` operator, the `all` operator effectively creates a constant. This semantics is similar to the proof technique of a for-all goal in first-order logic: to prove $\forall x. P$, we fix a constant $x$ and then prove $P$.

Consider the next example that synthesizes the identity function using the `all` operator.

```prolog
> (run 1 f
   (all (x)
     (== x (app f x))))
'((tie (_0) _0))
```

The implementation of the `all` operator follows that of the `fresh` operator, except that the created variable is a constant. In our implementation, we create an `all` variable as a free binding variable. Thus, the `all` variable cannot be unified with anything but itself.

5 λKANREN AS A THEOREM PROVER

λProlog is often regarded as a proof system. With hohh, λKanren suits a theorem prover as well. This section shows examples that use λKanren to prove intuitionistic style theorems.

We start with the definition of `proved`. At this moment, only `'trivial` is proved.

```prolog
(defrel (proved x)
  (== x 'trivial))
```
Obviously, not everything is proved.

\[
> \text{(run 1 g)}
> \quad \text{(all (p))}
> \quad \text{(proved p))}
> \quad \text{'}()
> \text{(run 1 g)}
> \quad \text{(proved g))}
> \quad \text{'}(\text{trivial})
\]

Next, we prove the commutativity of conjunction. I.e., \(\forall p, q (p \land q) \supset (q \land p)\).

\[
> \text{(run 1 g)}
> \quad \text{(all (p q))}
> \quad \text{(assume ((proved p) (proved q))}
> \quad \text{(proved q)}
> \quad \text{(proved p))})}
> \quad \text{'}(_0)
\]

The introduction rule of disjunction can be proved: \(\forall p, q p \supset (p \lor q)\)

\[
\text{(run 1 goal)}
> \quad \text{(all (p q))}
> \quad \text{(assume ((proved p))}
> \quad \text{(conde}
> \quad \quad \text{[(proved p)]}
> \quad \quad \text{[(proved q)])})}
> \quad \text{'}(_0)
\]

6 CONCLUSION

\lambda\text{Kanren} is based on higher-order hereditary Harrop formulas. It extends \mu\text{Kanren} with four operators, tie, app, assume-rel, and all. In addition, unification (\(=\)) identifies \(\alpha\beta\)-equivalence between the binding operators.

Our implementation of \lambda\text{Kanren} is written in Racket by adding about 40 lines to \mu\text{Kanren}. Overall, we appreciate the simplicity provided by the shallow embedding techniques.

REFERENCES


